

Perturbations of elliptic operators in chord arc domains.

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Abstract

We study the boundary regularity of solutions to divergence form operators which are small perturbations of operators for which the boundary regularity of solutions is known. An operator is a small perturbation of another operator if the deviation function of the coefficients satisfies a Carleson measure condition with small norm. We extend Escauriaza's result on Lipschitz domains to chord arc domains with small constant. In particular we prove that if L_1 is a small perturbation of L_0 and $\log k_0$ has small BMO norm so does $\log k_1$. Here k_i denotes the density of the elliptic measure of L_i with respect to the surface measure of the boundary of the domain.

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1 Introduction

In this paper we study the regularity properties of the elliptic measure associated to an elliptic operator in divergence form, $L = \operatorname{div} A \nabla$ on chord arc domains (CADs). We assume that A is a small perturbation of the matrix associate to a *regular* operator. See discussion below for the definition of small perturbation and the notion of *regular* operator. Chord arc domains are not necessary Lipschitz domains, in general they cannot be locally represented as graphs. This lack of a *preferred* direction even at the local level introduces a new set of challenges. On the other hand their geometry is sufficiently under control in order to develop and use tools from harmonic analysis. Chord arc domains in \mathbb{R}^n are non-tangentially accessible (NTA) domains whose boundaries are Ahlfors regular (a "non- degeneracy" condition indicating that the surface measure of $(n - 1)$ -dimensional balls with center on the boundary and radius r should behave like r^{n-1}). CADs are sets of locally finite perimeter (see [6]). In [13], Kenig and Toro showed that if Ω is a (δ, R) - chord arc domain with small δ (see Definition 2.7 below), then the unit normal to $\partial\Omega$ has small BMO constant with respect to $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ the surface measure to $\partial\Omega$.

For the Laplace operator, $L = \Delta$, Dahlberg [3] proved that if Ω is a strongly Lipschitz domain then the harmonic measure and the surface measure are mutually absolutely continuous and the Poisson kernel is in $L^2(\sigma)$. In [11], Jerison and Kenig showed that if Ω is a C^1 domain then $\log k$ (the logarithm of the Poisson kernel) belongs to $VMO(\sigma)$ where VMO is the Sarason space of vanishing mean oscillation. In [13], Kenig and Toro extended this result to a non-smooth setting by proving that if Ω is a chord arc domain with vanishing constant (see Definition 2.8 below) then $\log k$ belongs to $VMO(\sigma)$.

Questions concerning the regularity of the elliptic measure for variable coefficients operators in divergence form are rather delicate as was shown by the work of [1] and [16] where examples of operators with singular elliptic measures with respect to surface measure on smooth domains were constructed. Regularity results have been obtained, on Lipschitz domains, provided that the coefficient matrix A is given as a perturbation of a given matrix A_0 that corresponds to an elliptic operator whose elliptic measure is *regular* with respect to the surface measure to the boundary. In [4], Dahlberg introduced the notion of perturbation of elliptic operators in Lipschitz domains. Roughly speaking an operator $L = \operatorname{div} A \nabla$ is a perturbation of an operator $L_0 = \operatorname{div} A_0 \nabla$, if the difference between the coefficient matrices A and A_0 satisfies a Carleson condition.

More precisely, let $\Omega \subset \mathbb{R}^n$ be a CAD (see Definition 2.6) and consider two elliptic operators $L_i = \operatorname{div}(A_i \nabla)$ for $i = 0, 1$ in Ω . We say that L_1 is a perturbation of L_0 if the deviation function

$$a(X) = \sup\{|A_1(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\} \quad (1.1)$$

where $\delta(X)$ is the distance of X to $\partial\Omega$, satisfies the following Carleson measure property: there exists a constant $C > 0$ such that

$$\sup_{0 < r < \operatorname{diam}\Omega} \sup_{Q \in \partial\Omega} \left\{ \frac{1}{\sigma(B(Q, r))} \int_{B(Q, r) \cap \Omega} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq C. \quad (1.2)$$

Note that in this case $L_1 = L_0$ on $\partial\Omega$. L_1 is said to be a perturbation of L_0 with vanishing Carleson constant if for each compact $K \subset \mathbb{R}^n$

$$\lim_{r \rightarrow 0} \gamma_K(r) = 0 \quad (1.3)$$

where

$$\gamma_K(r) = \sup_{Q \in \partial\Omega \cap K} \sup_{0 < s \leq r} \left(\frac{1}{\sigma(B(Q, s))} \int_{B(Q, s) \cap \Omega} \frac{a^2(X)}{\delta(X)} dX \right)^{1/2}. \quad (1.4)$$

For $i = 0, 1$ we denote by $G_i(X, Y)$ the Green's function of L_i in Ω with pole at X and by ω_i^X the corresponding elliptic measure. Since the results below are independent of the pole X to

simplify notation we denote by ω_i the elliptic measure of L_i . Recall that k_i is the Radon-Nikodym derivative of ω_i with respect to σ .

Theorem 1.1 (Dahlberg [4]). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Assume that L_1 is a perturbation of L_0 with vanishing Carleson constant then $\omega_0 \in B_p(\sigma)$ for some $p \in (1, \infty)$ if and only if $\omega_1 \in B_p(\sigma)$.*

Theorem 1.2 (Fefferman-Kenig-Pipher [7]). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Assume that L_1 is a perturbation of L_0 then $\omega_0 \in A_\infty(\sigma)$ if and only if $\omega_1 \in A_\infty(\sigma)$.*

Theorem 1.3 (Escauriaza [5]). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Assume that L_1 is a perturbation of L_0 with vanishing Carleson constant then $\log k_0 \in VMO(\sigma)$ if and only if $\log k_1 \in VMO(\sigma)$.*

Dahlberg's proof is based on a very original idea. He shows that a differential inequality holds for a quantity that controls the $B_p(\sigma)$ norms of the elliptic measures a one parameter family of operators that interpolate between L_0 and L_1 . Escauriaza builds on this idea. On the other hand Fefferman, Kenig and Pipher use a completely different approach based on harmonic analysis techniques.

In this paper we extend Escauriaza's result to the CAD setting. The present work is a natural continuation of [15] and [14]. Although we follow Escauriaza's road map, the justification of most steps depend on arguments that resemble those of [14] which required developing harmonic analysis techniques on CADs. The recurrent theme is that since CAD are not locally representable as the graph of a good function we need to appeal to their geometry and the Ahlfors regality property of their boundary. In §2 we summarize some of the results from [14] and combine them with classical results from the theory of weights. In particular Corollary 2.15 guarantees that we can proceed as in Escauriaza's (see Remark 3.1). In §3 we prove the main result which reduces to a differential inequality which yields as in Dahlberg's and Escauriaza's case a bound for the appropriate $B_p(\sigma)$ norm.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider elliptic operators L of the form $Lu = \operatorname{div}(A(X)\nabla u)$ defined in Ω where $A(X) = (a_{ij}(X))$ is a symmetric matrix such that there are $\lambda, \Lambda > 0$ satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } X \in \Omega \text{ and } \xi \in \mathbb{R}^n. \quad (2.1)$$

We say that a function u in Ω is a solution to $Lu = 0$ in Ω provided that $u \in W_{\text{loc}}^{1,2}(\Omega)$ and for all $\phi \in C_c^\infty(\Omega)$, $\int_\Omega \langle A(x) \nabla u, \nabla \phi \rangle dx = 0$. A domain Ω is called regular for the operator L , if for every $g \in C(\partial\Omega)$, the generalized solution of the classical Dirichlet problem with boundary data g is a function $u \in C(\overline{\Omega})$. Let Ω be a regular domain for L , the Riesz Representation Theorem ensures that there exists a family of regular Borel probability measures $\{\omega_L^X\}_{X \in \Omega}$ such that the function $u(X) = \int_{\partial\Omega} g(Q) d\omega_L^X(Q)$ satisfies the Dirichlet problem for L with boundary data g . For $X \in \Omega$, ω_L^X is called the L -elliptic measure of Ω with pole X . When no confusion arises, we will omit the reference to L and simply call it as the elliptic measure.

The following lemmas contain some properties on the boundary behavior of L -elliptic solutions in non-tangentially accessible (NTA) domains for uniformly elliptic divergence form operators L with bounded measurable coefficients. We refer the reader to [10], [12] for the definitions and more details regarding elliptic operators of divergence form defined in NTA domains.

Lemma 2.1 (Cacciopoli Inequality). *Let u be a non-negative subsolution in Ω and $\overline{B(X, 2R)} \subset \Omega$. Then*

$$\int_{B(X, R)} |\nabla u(X)|^2 dX \leq \frac{C}{R^2} \int_{B(X, 2R)} u^2(X) dX$$

where constant C depends on the ellipticity constants λ, Λ and the dimension n .

Lemma 2.2 (Boundary Cacciopoli Inequality). *Let Ω be an NTA domain and $Q \in \partial\Omega$. If u satisfies $Lu = 0$ in $T(Q, 4R) = B(Q, 4R) \cap \Omega$ and $u = 0$ on $\Delta(Q, 4R) = B(Q, 4R) \cap \partial\Omega$ then*

$$\int_{T(Q, R)} |\nabla u(X)|^2 dX \leq \frac{C}{R^2} \int_{T(Q, 2R)} u^2(X) dX$$

where constant C depends on the ellipticity constants λ, Λ and the dimension n .

Lemma 2.3. *Let Ω be an NTA domain, $Q \in \partial\Omega$, $0 < 2r < R$, and $X \in \Omega \setminus B(Q, 2r)$. Then*

$$C^{-1} < \frac{\omega^X(B(Q, r))}{r^{n-2} G(A(Q, r), X)} < C,$$

where $G(A(Q, r), X)$ is the L -Green function of Ω with pole X , ω^X is the corresponding elliptic measure and $A(Q, r)$ is a non-tangential point for Q at r .

Lemma 2.4 (Comparison Principle). *Let Ω be an NTA domain and let $M > 1$ be such that $0 < Mr < R$. Suppose that u, v vanish continuously on $\partial\Omega \cap B(Q, Mr)$ for some $Q \in \partial\Omega$, $u, v \geq 0$ and $Lu = Lv = 0$ in $\Omega \cap B(Q, Mr)$. Then for all $X \in B(Q, r) \cap \Omega$,*

$$C^{-1} \frac{u(A(Q, r))}{v(A(Q, r))} \leq \frac{u(X)}{v(X)} \leq C \frac{u(A(Q, r))}{v(A(Q, r))}$$

where the constant $C > 1$ only depends on the dimension, the NTA constants and the ellipticity constants.

An immediate consequence of the previous lemma is the following boundary regularity result.

Lemma 2.5 (Hölder Regularity). *Let u, v be as in Lemma 2.4, then there exists $\vartheta \in (0, 1)$ such that*

$$\left| \frac{u(Y)}{v(Y)} - \frac{u(X)}{v(X)} \right| \leq \frac{u(A_r(Q))}{v(A_r(Q))} \left(\frac{|X - Y|}{r} \right)^{\vartheta}$$

for all $X, Y \in B(Q, r) \cap \Omega$. Here ϑ depends on the dimension, the NTA constants and the ellipticity constants.

Definition 2.6. We say that $\Omega \subset \mathbb{R}^n$ is a chord arc domain (CAD) if Ω is an NTA domain whose boundary is Ahlfors regular, i.e. the surface measure to the boundary satisfies the following condition: there exists $C > 1$ so that for $r \in (0, \text{diam } \Omega)$ and $Q \in \partial\Omega$

$$C^{-1}r^{n-1} \leq \sigma(B(Q, r)) \leq Cr^{n-1}. \quad (2.2)$$

Here $B(Q, r)$ denotes the n -dimensional ball of radius r and center Q and $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ and \mathcal{H}^{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure. The best constant C above is referred to as the Ahlfors regularity constant.

As mentioned earlier CAD are sets of locally finite perimeter (see [6]). Let $\Omega \subset \mathbb{R}^n$ be a domain. Let D denote Hausdorff distance between closed sets. We define

$$\theta(r) = \sup_{Q \in \partial\Omega} \inf_{\mathcal{L}} r^{-1} D[\partial\Omega \cap B(Q, r), \mathcal{L} \cap B(Q, r)], \quad (2.3)$$

where the infimum is taken over all $(n-1)$ -planes containing $Q \in \partial\Omega$.

Definition 2.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\delta > 0$ and $R > 0$. We say that Ω is a (δ, R) -chord arc domain (CAD) if Ω is a set of locally finite perimeter such that

$$\sup_{0 < r < R} \theta(r) \leq \delta \quad (2.4)$$

and

$$\sigma(B(Q, r)) \leq (1 + \delta)\omega_{n-1}r^{n-1} \quad \forall Q \in \partial\Omega \quad \text{and} \quad \forall r \in (0, R). \quad (2.5)$$

Here ω_{n-1} is the volume of the $(n-1)$ -dimensional unit ball in \mathbb{R}^{n-1} .

Definition 2.8. Let $\Omega \subset \mathbb{R}^n$, we say that Ω is a chord arc domain with vanishing constant if it is a (δ, R) -CAD for some $\delta > 0$ and $R > 0$,

$$\limsup_{r \rightarrow 0} \theta(r) = 0 \quad (2.6)$$

and

$$\lim_{r \rightarrow 0} \sup_{Q \in \partial\Omega} \frac{\sigma(B(Q, r))}{\omega_n r^{n-1}} = 1. \quad (2.7)$$

Next we recall some fine properties concerning perturbations of elliptic operators in CAD (see (1.2), (1.3) (1.4) for the relevant definitions). In [14], we showed that we may assume $a(X) = 0$ in Ω for $X \in \Omega$ with $\delta(X) > 4R_0$ where $R_0 = \frac{1}{2^{30}} \min\{\delta(0), 1\}$ and $0 \in \Omega$. In particular we cover the boundary $\partial\Omega$ by balls $\{B(Q_i, R_0/2)\}_{i=1}^M$ such that $Q_i \in \partial\Omega$ and $|Q_i - Q_j| \geq \frac{R_0}{2}$ for $i \neq j$ and consider the partition of unity $\{\varphi_i\}_{i=1}^M$ associated with this covering that satisfies $0 \leq \varphi_i \leq 1$, $\text{spt}\varphi_i \subset B(Q_i, 2R_0)$, $\varphi_i \equiv 1$ on $B(Q_i, R_0)$ and $|\nabla\varphi_i| \leq 4/R_0$. Then if we define

$$\psi_i(X) = \begin{cases} \left(\sum_{j=1}^M \varphi_j(X) \right)^{-1} \varphi_i(X) & \text{if } \sum_{j=1}^M \varphi_j(X) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$A'(X) = \left(\sum_{j=1}^M \psi_j(X) \right) A_1(X) + \left(1 - \sum_{j=1}^M \psi_j(X) \right) A_0(X) \quad (2.8)$$

the following lemmas hold.

Lemma 2.9 ([14]). Let A' be as in (2.8) then for $X \in \Omega$, with $\delta(X) > 4R_0$,

$$a'(X) = \sup_{B(X, \delta(X)/2)} |A'(Y) - A_0(Y)| = 0.$$

Lemma 2.10 ([14]). If ω' denotes the elliptic measure associated to $L' = \text{div} A' \nabla$ with pole at 0, then $\omega_1 \in B_p(\omega_0)$ if and only if $\omega' \in B_p(\omega_0)$. Here we assume that both ω_0 and ω_1 have pole at 0.

One of the main results in [14] concerns the regularity of the elliptic measure of perturbation operators in CADs. In particular it was shown that if a Carleson norm of the deviation function (see 1.1) is small then "good" properties of the elliptic measure are preserved.

Theorem 2.11 ([14]). Let Ω be a CAD, $0 \in \Omega$ and ω_0, ω_1 are the elliptic measures associated with L_0 and L_1 respectively with pole 0. There exists $\varepsilon_0 > 0$, depending also on the ellipticity constants, the dimension, the CAD constants such that if

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \varepsilon_0 \quad \text{then} \quad \omega_1 \in B_2(\omega_0). \quad (2.9)$$

Here $T(\Delta) = B(Q, r) \cap \Omega$ is the tent associated to the surface ball $\Delta = \Delta_r(Q) = B(Q, r) \cap \partial\Omega$ and $G_0(X) = G_0(0, X)$ denotes the Green's function for L_0 in Ω with pole at 0.

Note that (2.9) and the Carleson measure property (1.2) relate as follows.

Proposition 2.12 ([14]). *Let Ω be a CAD and that assume $\omega_0 \in B_p(\sigma)$ for some $p > 1$. Given $\epsilon > 0$ there exists $\delta > 0$ such that if*

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq \delta, \quad (2.10)$$

then

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon. \quad (2.11)$$

An immediate consequence of Theorem 2.11 deals with the $L^r(d\sigma)$ -integrability of $k_1 = \frac{d\omega_1}{d\sigma}$ provided that a suitable condition is assumed for ω_0 .

Theorem 2.13 ([14]). *Let Ω be a CAD and ω_0, ω_1 be as in Theorem 2.11. If $\omega_1 \in B_p(\omega_0)$ for some $1 < p < \infty$ and $\omega_0 \in B_q(\sigma)$ then $\omega_1 \in B_r(\sigma)$ with $r = \frac{qp}{q+p-1} < q$.*

Proof. Consider $r = \frac{qp}{q+p-1}$ and let $h = d\omega_1/d\omega_0$, $k_0 = d\omega_0/d\sigma$ and $k_1 = d\omega_1/d\sigma$. Then

$$\int_{\Delta} k_1^r d\sigma = \int_{\Delta} h^r k_0^{r/p} k_0^{r(1-1/p)} d\sigma \leq \left(\int_{\Delta} (h^r k_0^{r/p})^{q/(q-(1-1/p)r)} d\sigma \right)^{\frac{q-(1-1/p)r}{q}} \left(\int_{\Delta} k_0^q d\sigma \right)^{\frac{r(1-1/p)}{q}}$$

that is,

$$\int_{\Delta} k_1^r d\sigma \leq \left(\int_{\Delta} h^p d\omega_0 \right)^{\frac{q-(1-1/p)r}{q}} \left(\int_{\Delta} k_0^q d\sigma \right)^{\frac{r}{q(1-1/p)}}$$

or by the selection of r ,

$$\int_{\Delta} k_1^r d\sigma \leq \left(\int_{\Delta} h^p d\omega_0 \right)^{\frac{q}{q+p-1}} \left(\int_{\Delta} k_0^q d\sigma \right)^{\frac{p-1}{p+q-1}}.$$

Since

$$\int_{\Delta} k_0^q d\sigma \leq \sigma(\Delta) \left(\int_{\Delta} k_0 d\sigma \right)^q \quad \text{and} \quad \int_{\Delta} h^p d\omega_0 \leq \omega_0(\Delta) \left(\int_{\Delta} h d\omega_0 \right)^p$$

we conclude that

$$\int_{\Delta} k_1^r d\sigma \lesssim \left(\int_{\Delta} k_1 d\sigma \right)^r \sigma(\Delta)^{1-r}$$

or

$$\left(\int_{\Delta} k_1^r d\sigma \right)^{1/r} \lesssim \int_{\Delta} k_1 d\sigma$$

and the proof is complete since $r = \frac{qp}{q+p-1} < q$. □

Throughout the paper we shall use the notation $a \lesssim b$ to mean that there is a constant $C > 0$ such that $a \leq Cb$.

A slight improvement of the result in Theorem 2.11 can be obtained due to an argument of Gehring ([8], Lemma 2), see also the book of Grafakos ([9]).

Lemma 2.14. *Let Ω be a CAD and ω_0, ω_1 be as in Theorem 2.11. If condition (2.9) is satisfied then there exists a constant $\eta_0 > 0$ which depends only on the constant ε_0 which appears in (2.9), the CAD and ellipticity constants such that $\omega_1 \in B_{2(1+\eta_0)}(\omega_0)$.*

Once we combine Theorem 2.13 along with Lemma 2.14 we obtain the following corollary.

Corollary 2.15. *Let Ω be a CAD and ω_0, ω_1 be as in Theorem 2.11. For $\delta_0 > 0$ small enough there exists q_0 large enough depending only on the CAD constants, the dimension and the ellipticity constants such that if $\omega_0 \in B_{q_0}(\sigma)$ then $\omega_1 \in B_{2(1+\delta_0)}(\sigma)$.*

In the sequel we denote the area integral and the nontangential maximal function by

$$S_M(u)(Q) = \left(\int_{\Gamma_M(Q)} |\nabla u(X)|^2 \delta(X)^{2-n} dX \right)^{1/2} \quad \text{and} \quad N(u)(Q) = \sup\{|u(X)| : X \in \Gamma_M(Q)\}$$

respectively where for $Q \in \partial\Omega$

$$\Gamma_M(Q) = \{X \in \Omega : |X - Q| < (1 + M)\delta(X)\}. \quad (2.12)$$

The following lemma will be used in Section 3.

Lemma 2.16 ([12]). *Let $\mu \in A_\infty(d\omega)$, $0 \in \Omega$. Then if $Lu = 0$ and $0 < p < \infty$,*

$$\left(\int_{\partial\Omega} (S_\alpha(u))^p d\mu \right)^{1/p} \leq C_{\alpha,p} \left(\int_{\partial\Omega} (N_\alpha(u))^p d\mu \right)^{1/p}.$$

If in addition $u(0) = 0$ then

$$\left(\int_{\partial\Omega} (N_\alpha(u))^p d\mu \right)^{1/p} \leq C_{\alpha,p} \left(\int_{\partial\Omega} (S_\alpha(u))^p d\mu \right)^{1/p}.$$

Suppose also that f is a measurable function defined in Ω . For $\alpha > 0$ and $Q \in \partial\Omega$, we define

$$A^{(\alpha)}(f)(Q) = \left(\int_{\Gamma_\alpha(Q)} f(X)^2 \frac{dX}{\delta(X)^n} \right)^{1/2}. \quad (2.13)$$

The usual square function of f corresponds to $A(f) = A^{(1)}(f)$. We define the operator $\mathcal{C}(f) : \partial\Omega \rightarrow \mathbb{R}$ by

$$\mathcal{C}(f)(Q) := \sup_{Q \in \Delta} \left(\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} f(X)^2 \frac{dX}{\delta(X)} \right)^{1/2} \quad (2.14)$$

where Δ is a surface ball and $T(\Delta)$ is the tent over it.

In the present paper we use the same family of dyadic cubes in $\partial\mathcal{O}$ as the one used in [14]. The *shadows* of the dyadic cubes in \mathcal{O} provide a good covering of $\mathcal{O} \cap (\partial\mathcal{O}, 4R_0) := \mathcal{O} \cap \{Y \in \mathbb{R}^n : \exists Q_Y \in \partial\Omega \text{ with } |Q_Y - Y| = \delta(Y) \leq 4R_0\}$. To ease the readers task we recall some of their main properties. Since Ω is a CAD in \mathbb{R}^n , both $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ and ω_0 are doubling measures and therefore $(\partial\Omega, |\cdot|, \sigma)$ and $(\partial\Omega, |\cdot|, \omega_0)$ are spaces of homogeneous type. M. Christ's construction (see [2]) ensures that there exists a family of dyadic cubes $\{Q_\alpha^k \subset \partial\Omega : k \in \mathbb{Z}, \alpha \in I_k\}$, $I_k \subset \mathbb{N}$ such that for every $k \in \mathbb{Z}$

$$\sigma(\partial\Omega \setminus \bigcup_{\alpha} Q_\alpha^k) = 0, \quad \omega_0(\partial\Omega \setminus \bigcup_{\alpha} Q_\alpha^k) = 0. \quad (2.15)$$

and the following properties are satisfied:

1. If $l \geq k$ then either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$.
2. For each (k, α) and each $l < k$ there is a unique β so that $Q_\alpha^k \subset Q_\beta^l$.
3. There exists a constant $C_0 > 0$ such that $\text{diam } Q_\alpha^k \leq C_0 8^{-k}$.
4. Each Q_α^k contains a ball $B(Z_\alpha^k, 8^{-k-1})$.

The Ahlfors regularity property of σ combined with properties 3 and 4 ensure that there exists $C_1 > 1$ such that

$$C_1^{-1} 8^{-k(n-1)} \leq \sigma(Q_\alpha^k) \leq C_1 8^{-k(n-1)}. \quad (2.16)$$

In addition the doubling property of ω_0 yields

$$\omega_0(B(Z_\alpha^k, 8^{-k-1})) \sim \omega_0(Q_\alpha^k). \quad (2.17)$$

For $k \in \mathbb{Z}$ and $\alpha \in I_k$ we define

$$I_\alpha^k = \{Y \in \Omega : \lambda 8^{-k-1} < \delta(Y) < \lambda 8^{-k+1}, \exists P \in Q_\alpha^k \text{ so that } \lambda 8^{-k-1} < |P - Y| < \lambda 8^{-k+1}\}, \quad (2.18)$$

where $\lambda > 0$ is chosen so that for each k , the $\{I_\alpha^k\}_{\alpha \in I_k}$'s have finite overlaps and

$$\Omega \cap (\partial\Omega, 4R_0) \subset \bigcup_{k \leq k_0, \alpha} I_\alpha^k. \quad (2.19)$$

Here k_0 can be selected so that $4R_0 < \lambda 8^{-k_0-1}$. We refer the reader to [14] for the proof of (2.19) and the details on the construction of $\{Q_\alpha^k\}$ and $\{I_\alpha^k\}$.

The various constants that will appear in the sequel may vary from formula to formula, although for simplicity we use the same letter. If we do not give any explicit dependence for a constant, we mean that it depends only on the ellipticity constants, CAD constants and the dimension.

3 Main Result

In this section we state and prove the main result of the present work. Assume that $L_0 = \operatorname{div}(A_0 \nabla)$ and $L_1 = \operatorname{div}(A_1 \nabla)$ are two symmetric divergence form operators satisfying (2.1) defined in a CAD Ω containing 0. We denote the deviation function of L_1 from L_0 by

$$a(X) = \sup\{|A_1(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\}$$

and we assume that L_1 is a perturbation of L_0 . For $t \in [0, 1]$ we consider the operators defined by

$$L_t u = \operatorname{div}(A_t \nabla u) \quad (3.1)$$

$$A_t(X) = (1-t)A_0(X) + tA_1(X). \quad (3.2)$$

Note that for each t , L_t satisfies (2.1). Let ω_t be the corresponding L_t -elliptic measure with pole 0 and let $G_t(0, Y)$ be the Green's function for L_t .

Remark 3.1. Note that since

$$a_t(X) = \sup\{|A_t(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\} = ta(X)$$

then L_t is also a perturbation of L_0 . Moreover under the assumptions of Corollary 2.15, we have that for every $t \in [0, 1]$ ω_t is a $B_{2(1+\delta_0)}(\sigma)$ -weight with a uniform $B_{2(1+\delta_0)}$ -constant. Thus in particular for $t \in [0, 1]$, $\omega_t \in B_2(\sigma)$. From now on we assume that $\mathcal{C}(a)$ is small enough so that the hypothesis of Theorem 2.11 and those of Corollary 2.15 are satisfied (see Proposition 2.12).

We consider the Dirichlet problems

$$\begin{cases} L_t u_t = 0 & \text{in } \Omega \\ u_t = f & \text{on } \partial\Omega \end{cases} \quad \begin{cases} L_s u_s = 0 & \text{in } \Omega \\ u_s = f & \text{on } \partial\Omega \end{cases} \quad (3.3)$$

for $s, t \in [0, 1]$, where $f \in L^2(\sigma)$.

Lemma 3.2. *Let Ω be a CAD, $0 \in \Omega$. Under the assumptions in Remark 3.1, if u_t, u_s are solutions to the Dirichlet problems (3.3) then*

$$u_s(0) - u_t(0) = (s - t) \int_{\Omega} \varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y) dY \quad (3.4)$$

and

$$\int_{\Omega} |\varepsilon(Y)| |\nabla G_t(0, Y)| |\nabla u_s(Y)| dY \lesssim \|f\|_{L^2(\sigma)}. \quad (3.5)$$

In particular

$$|u_s(0) - u_t(0)| \lesssim \|f\|_{L^2(\sigma)} |s - t|. \quad (3.6)$$

Proof. Assume that $\delta(0) = 4R_0$. Without loss of generality we assume that $A_0 = A_1$ on $B(0, R_0)$ and $s > t$. Then integration by parts shows that

$$u_s(0) - u_t(0) = \int_{\Omega} G_t(0, Y) L_t u_s(Y) dY = (s - t) \int_{\Omega} \varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y) dY \quad (3.7)$$

which proves (3.4). To prove (3.5) we proceed as in the proof of Lemma 7.7 in [14] using the dyadic surface cubes and their interior *shadows* described in Section 2. Assume that $\Omega \setminus (\Omega, R_0) \subset \bigcup_i B(Q_i, 2R_0) \cap \Omega$ where $|Q_i - Q_j| \geq R_0$, $Q_i \in \partial\Omega$. Note that the family of balls has finite overlap. First we estimate the integral in the tent over $\Delta_0 = B(Q_i, 2R_0) \cap \partial\Omega$.

$$\int_{B(Q_i, 2R_0) \cap \Omega} |\varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y)| dY = \lim_{\delta \rightarrow 0} \int_{T(\Delta_0) \setminus (\partial\Omega, \delta)} |\varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y)| dY$$

where $T(\Delta_0) = B(Q_i, 2R_0) \cap \Omega$. For $\delta > 0$ small we compute

$$I_1 = \int_{T(\Delta_0) \setminus (\partial\Omega, \delta)} |\varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y)| dY \leq \sum_{\substack{Q_{\alpha}^k \subset 3\Delta_0 \\ \delta < \lambda 8^{-k-1}}} \sup_{I_{\alpha}^k} |\varepsilon(Y)| \int_{I_{\alpha}^k} |\nabla G_t(0, Y)| |\nabla u_s(Y)| dY \quad (3.8)$$

and for $Y \in I_{\alpha}^k$

$$|\nabla G_t(0, Y)| \lesssim \frac{G_t(0, Y)}{\delta(Y)} \sim \frac{\omega_t(Q_{\alpha}^k)}{(\text{diam } Q_{\alpha}^k)^{n-1}}$$

thus

$$\begin{aligned} I_1 &\lesssim \sum_{\substack{Q_{\alpha}^k \subset 3\Delta_0 \\ \delta < \lambda 8^{-k-1}}} \sup_{I_{\alpha}^k} |\varepsilon(Y)| \frac{\omega_t(Q_{\alpha}^k)}{(\text{diam } Q_{\alpha}^k)^{n-1}} \int_{I_{\alpha}^k} |\nabla u_s(Y)| dY \\ &\lesssim \sum_{\substack{Q_{\alpha}^k \subset 3\Delta_0 \\ \delta < \lambda 8^{-k-1}}} \sup_{I_{\alpha}^k} |\varepsilon(Y)| \frac{\omega_t(Q_{\alpha}^k)}{(\text{diam } Q_{\alpha}^k)^{n-1}} \left(\int_{I_{\alpha}^k} |\nabla u_s(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2} (\text{diam } Q_{\alpha}^k)^{n/2} (\text{diam } Q_{\alpha}^k)^{n/2-1} \\ &\lesssim \sum_{\substack{Q_{\alpha}^k \subset 3\Delta_0 \\ \delta < \lambda 8^{-k-1}}} \sup_{I_{\alpha}^k} |\varepsilon(Y)| \frac{\omega_t(Q_{\alpha}^k)}{(\text{diam } Q_{\alpha}^k)^{n-1}} \left(\int_{I_{\alpha}^k} |\nabla u_s(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2} (\text{diam } Q_{\alpha}^k)^{n-1} \\ &\lesssim \sum_{\substack{Q_{\alpha}^k \subset 3\Delta_0 \\ \delta < \lambda 8^{-k-1}}} \left(\int_{I_{\alpha}^k} \frac{a^2(Y)}{\delta(Y)^n} \cdot \frac{\omega_t(Q_{\alpha}^k)^2}{(\text{diam } Q_{\alpha}^k)^{2n-2}} dY \right)^{1/2} \left(\int_{I_{\alpha}^k} |\nabla u_s(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2} (\text{diam } Q_{\alpha}^k)^{n-1} \\ &\lesssim \int_{3\Delta_0} \left(\sum \int_{I_{\alpha}^k} \frac{a^2(Y)}{\delta(Y)^n} \cdot \frac{\omega_t(B(Q_Y, \delta(Y)))^2}{\delta(Y)^{2n-2}} dY \chi_{Q_{\alpha}^k(Q)} \right)^{1/2} \left(\sum \int_{I_{\alpha}^k} |\nabla u_s(Y)|^2 \delta(Y)^{2-n} dY \chi_{Q_{\alpha}^k(Q)} \right)^{1/2} d\sigma \\ &\lesssim \int_{3\Delta_0} \left(\int_{\Gamma_M(Q)} \frac{a^2(Y)}{\delta(Y)^n} H_t(Y)^2 dY \right)^{1/2} S_M(u_s) d\sigma(Q) \\ &\lesssim \left(\int_{3\Delta_0} \int_{\Gamma_M(Q)} \frac{a^2(Y)}{\delta(Y)^n} H_t(Y)^2 d\sigma \right)^{1/2} \left(\int_{3\Delta_0} S_M(u_s)^2 d\sigma \right)^{1/2} \end{aligned} \quad (3.9)$$

where $H_t(Y) = \frac{\omega_t(B(Q_Y, \delta(Y)))}{\delta(Y)^{n-1}}$. Since $\omega_s \in B_2(\sigma)$ by Remark 3.1 applying Lemma 2.16 for $p = 2$ and recalling that the $L^2(\sigma)$ norm of the non-tangential maximal function of u_s is bounded by the $L^2(\sigma)$ norm of f we obtain

$$I_1 \lesssim \left(\int_{3\Delta_0} \int_{\Gamma_M(Q)} \frac{a^2(Y)}{\delta(Y)^n} H_t(Y)^2 d\sigma \right)^{1/2} \|f\|_{L^2(\sigma)} := D \|f\|_{L^2(\sigma)}. \quad (3.10)$$

We now estimate D :

$$\begin{aligned} D^2 &= \int_{3\Delta_0} \int_{\Gamma_M(Q)} \frac{a^2(Y)}{\delta(Y)^n} H_t(Y)^2 dY d\sigma = \int_{3\Delta_0} \int_{\Omega} \chi_{\Gamma_M(Q)}(Y) \frac{a^2(Y)}{\delta(Y)^n} H_t(Y)^2 dY d\sigma \\ &\lesssim \int_{12\Delta_0} \left(\int_{\Gamma_M(Q)} \frac{a^2(Y)}{\delta(Y)^n} H_t(Y) dY \right) d\omega_t \\ &\lesssim \int_{12\Delta_0} N H_t(Q) A(a)(Q)^2 d\omega_t \\ &\lesssim \left(\int_{12\Delta_0} N H_t(Q)^p d\omega_t \right)^{1/p} \left(\int_{12\Delta_0} A(a)(Q)^{2q} d\omega_t \right)^{1/q} \end{aligned} \quad (3.11)$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p = 1 + \delta_0$ and δ_0 is selected as in Corollary 2.15. Note also that

$$\left(\int_{12\Delta_0} A(a)(Q)^{2q} k_t d\sigma \right)^{1/q} \leq \left(\int_{12\Delta_0} A(a)(Q)^{4q} d\sigma \right)^{1/2q} \left(\int_{12\Delta_0} k_t^2 d\sigma \right)^{1/2q}$$

where

$$\left(\int_{12\Delta_0} A(a)(Q)^{4q} d\sigma \right)^{1/2q} \lesssim \mathcal{C}(a)^2 \sigma(12\Delta_0)^{1/2q}$$

and $\mathcal{C}(a)$ is defined in (2.14). Therefore, if Mk_t denotes the maximal function

$$\begin{aligned} D^2 &\lesssim \left(\int_{12\Delta_0} k_t^2 d\sigma \right)^{1/2p} \left(\int_{12\Delta_0} M k_t^{2p} d\sigma \right)^{1/2p} \mathcal{C}(a) \sigma(12\Delta_0)^{1/2q} \left(\int_{12\Delta_0} k_t^2 d\sigma \right)^{1/2q} \\ &\lesssim \mathcal{C}(a) \left(\int_{12\Delta_0} k_t^2 d\sigma \right)^{1/2} \left(\int_{24\Delta_0} k_t^{2p} d\sigma \right)^{1/2p} \sigma(12\Delta_0)^{1/2q} \sigma(24\Delta_0)^{1/2+1/2p} \\ &\lesssim \mathcal{C}(a) \left(\int_{12\Delta_0} k_t d\sigma \right) \left(\int_{12\Delta_0} k_t d\sigma \right) \sigma(12\Delta_0) \\ &\lesssim \mathcal{C}(a)^2 \frac{\omega_t(12\Delta_0)^2}{\sigma(12\Delta_0)}. \end{aligned} \quad (3.12)$$

Note that the estimate for I_1 is independent of δ and therefore we conclude combining (3.8), (3.10), (3.12) that

$$\begin{aligned} \int_{B(Q_i, 2R_0) \cap \Omega} |\varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y)| dY &\lesssim \mathcal{C}(a) \frac{\omega_t(12\Delta_0)}{\sqrt{\sigma(12\Delta_0)}} \|f\|_{L^2(\sigma)} \\ &\lesssim \|f\|_{L^2(\sigma)}. \end{aligned} \quad (3.13)$$

We now estimate the integral over the complement of the tent in Ω .

$$\begin{aligned} I_2 &= \int_{\Omega \setminus T(\Delta_0)} |\varepsilon(Y) \nabla G_t(0, Y) \nabla u_s(Y)| dY \lesssim \int_{\partial\Omega \setminus \frac{1}{4}\Delta_0} S_M(u_s) d\omega_t \\ &\lesssim \left(\int_{\partial\Omega} S_M(u_s)^2 d\sigma \right)^{1/2} \left(\int_{\partial\Omega} k_t^2 d\sigma \right)^{1/2} \lesssim \|f\|_{L^2(\sigma)}. \end{aligned} \quad (3.14)$$

Since Ω is a bounded domain and $\omega_t \in B_2(\sigma)$ then the $L^2(\sigma)$ norm of k_t is bounded in terms of the $B_2(\sigma)$ norm of ω_t and R_0 . The proof is completed by combining (3.13) and (3.14). \square

Let $Q_0 \in \partial\Omega$ be fixed and $r > 0$. Let $\Delta_r = \Delta_r(Q_0) = \partial\Omega \cap B(Q_0, r)$ and $T(\Delta_r) = T(\Delta_r(Q_0)) = \Omega \cap B(Q_0, r)$.

Lemma 3.3. *Under the assumptions in Remark 3.1, for $f \in L^2(\sigma)$ and $t \in [0, 1]$ consider*

$$\Psi(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} f k_t d\sigma.$$

Then $\Psi(t)$ is Lipschitz. Moreover

$$\dot{\Psi}(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} \dot{k}_t \left(f - \oint_{\Delta_r} f d\omega_t \right) d\sigma$$

where \dot{k}_t exists as the weak L^2 limit of $k_{t+h} - k_t/h$ as h tends to zero.

Proof. Let $s, t \in [0, 1]$ and denote by u_t, u_s the solutions of (3.3) for a given $f \in L^2(\sigma)$. Then

$$\begin{aligned} \Psi(t) - \Psi(s) &= \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} k_t f d\sigma - \frac{1}{\omega_s(\Delta_r)} \int_{\Delta_r} k_s f d\sigma \\ &= \frac{1}{\omega_t(\Delta_r)} \left(\int_{\Delta_r} k_t f d\sigma - \int_{\Delta_r} k_s f d\sigma \right) + \left(\frac{1}{\omega_t(\Delta_r)} - \frac{1}{\omega_s(\Delta_r)} \right) \int_{\Delta_r} k_s f d\sigma \end{aligned}$$

which will show that Ψ is Lipschitz due to (3.6). In particular we have that

$$\left| \frac{u_{t+h} - u_t}{h} \right| = \left| \int_{\partial\Omega} \frac{k_{t+h} - k_t}{h} f d\sigma \right| \lesssim \|f\|_{L^2(\sigma)}$$

which shows that \dot{k}_t exists as a weak L^2 limit of $k_{t+h} - k_t/h$ as h tends to zero. In order to compute $\dot{\Psi}(t)$ we write

$$\begin{aligned} \frac{\Psi(t+h) - \Psi(t)}{h} &= \frac{1}{\omega_{t+h}(\Delta_r)} \int_{\Delta_r} \frac{k_{t+h} - k_t}{h} f d\sigma + \frac{1}{h} \left(\frac{1}{\omega_{t+h}(\Delta_r)} - \frac{1}{\omega_t(\Delta_r)} \right) \int_{\Delta_r} k_t f d\sigma \\ &= \frac{1}{\omega_{t+h}(\Delta_r)} \int_{\Delta_r} \frac{k_{t+h} - k_t}{h} f d\sigma + \frac{\omega_t(\Delta_r) - \omega_{t+h}(\Delta_r)}{h} \frac{1}{\omega_{t+h}(\Delta_r) \omega_t(\Delta_r)} \int_{\Delta_r} k_t f d\sigma \end{aligned}$$

thus

$$\dot{\Psi}(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} \dot{k}_t \left(f - \oint_{\Delta_r} f d\omega_t \right) d\sigma.$$

\square

Lemma 3.4. *Under the assumptions in Remark 3.1, there exist positive constants $\beta, \gamma < 1$ and C such that if $\Psi(t)$ is the function defined in Lemma 3.3 where f is a non-negative function with $\text{spt}(f) \subset \Delta_r(Q_0)$ and $\|f\|_{L^2(d\sigma/\sigma(\Delta(Q_0, r)))} \leq 1$ then*

$$|\dot{\Psi}(t)| \leq C \left[r^\gamma + \sup_{\substack{Q \in \partial\Omega \\ s \leq r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))} \int_{T(\Delta_s(Q))} \frac{a(Y)^2}{\delta(Y)} dY \right)^{1/2} \right] \quad (3.15)$$

for $0 \leq t \leq 1$.

Proof. Assume that u_t is the solution of the problem

$$\begin{cases} L_t u_t = 0 & \text{in } \Omega \\ u_t = h_t & \text{on } \partial\Omega \end{cases} \quad (3.16)$$

where

$$h_t = \frac{1}{\omega_t(\Delta_r)} (f - \int_{\Delta_r} f d\omega_t) \chi_{\Delta_r}$$

and $\Delta_r = \Delta_r(Q)$. As in (3.4) and Lemma 3.3 we have that

$$\dot{\Psi}(t) \leq \int_{\Omega} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY.$$

We prove Lemma 3.4 using the following three claims.

Claim 1. For fixed $Q_0 \in \partial\Omega$, $r > 0$ and $\Delta_{Mr} = \Delta_{Mr}(Q_0)$, $M > 0$

$$\int_{T(\Delta_{Mr})} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \lesssim \sup_{\substack{Q \in \partial\Omega \\ s \leq r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))} \int_{T(\Delta_s(Q))} \frac{a(Y)^2}{\delta(Y)} dY \right)^{1/2} \quad (3.17)$$

where β is a given positive constant.

Proof of Claim 1. To prove (3.17) we proceed as in the proof of (3.9) in Lemma 3.2. In a similar manner we obtain the analog of (3.10)-(3.13) which in this case yield

$$\int_{T(\Delta_{Mr})} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \lesssim C(a) \frac{\omega_t(\Delta_r)}{\sqrt{\sigma(\Delta_r)}} \|h_t\|_{L^2} \lesssim \sup_{\substack{Q \in \partial\Omega \\ s \leq r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))} \int_{T(\Delta_s(Q))} \frac{a(Y)^2}{\delta(Y)} dY \right)^{1/2}$$

due to the selection of the boundary data h_t .

Claim 2. Let $0 < r < 8R_0$ and R_0 is selected as in Lemma 3.2. For fixed $Q_0 \in \partial\Omega$, there exists a constant $\eta > 0$ such that

$$\int_{\Omega \setminus T(\Delta_{8R_0}) \cap (\partial\Omega, 4R_0)} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \lesssim r^\eta \quad (3.18)$$

where $\Delta_{8R_0} = \Delta_{8R_0}(Q_0)$.

Proof of Claim 2. Note that $\varepsilon(Y) \equiv 0$ in $B(0, \frac{\delta(0)}{4})$ where $G_t(0, -)$ denotes the Green's function of L_t with pole at 0. We denote by $\Gamma_{R_0} = \Omega \setminus T(\Delta_{8R_0}) \cap (\partial\Omega, 4R_0)$ and apply Schwartz's inequality to obtain

$$\int_{\Gamma_{R_0}} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \lesssim \sup_{(\partial\Omega, 4R_0)} |\varepsilon(Y)| \left(\int_{\Gamma_{R_0}} |\nabla u_t|^2 dY \right)^{1/2} \left(\int_{(\partial\Omega, 4R_0)} |\nabla_Y G_t|^2 dY \right)^{1/2}. \quad (3.19)$$

In addition,

$$\left(\int_{(\partial\Omega, 4R_0)} |\nabla_Y G_t|^2 dY \right)^{1/2} \lesssim R_0^{n-2/2} G_t(0, A_{R_0}(Q_0)) \lesssim R_0^{\frac{n-2}{2}} \frac{\omega(\Delta_{R_0}(Q_0))}{R_0^{n-2}} \lesssim R_0^{-\frac{n-2}{2}} \quad (3.20)$$

where $A_{R_0}(Q_0) = (1 - R_0)Q_0$. We will now estimate $\sup |u_t|$. Note that

$$u_t = \int_{\partial\Omega} h_t d\omega_t^X = \int_{\partial\Omega} h_t K_t(X, Q) d\omega_t$$

where $K_t(X, Q) = \frac{d\omega_t^X}{d\omega_t}(Q)$ and $d\omega_t = k_t d\sigma$. For $\frac{1}{4}|X - Q_0| > |Q - Q_0|$ we have

$$\begin{aligned} |K_t(X, Q) - K_t(X, Q_0)| &\lesssim \left(\frac{|Q - Q_0|}{|X - Q_0|} \right)^\eta \frac{G_t(X, A_{|X-Q_0|}(Q_0))}{G_t(0, A_{|X-Q_0|}(Q_0))} \\ &\lesssim \left(\frac{|Q - Q_0|}{|X - Q_0|} \right)^\eta \frac{\omega_t^X(\Delta_{|X-Q_0|}(Q_0))}{\omega_t(\Delta_{|X-Q_0|}(Q_0))} \\ &\lesssim \left(\frac{|Q - Q_0|}{|X - Q_0|} \right)^\eta \frac{1}{\omega_t(\Delta_{|X-Q_0|}(Q_0))} \end{aligned}$$

By its definition $h_t = 0$ outside Δ_r therefore,

$$u_t(X) = \int_{\partial\Omega} (K_t(X, Q) - K_t(X, Q_0)) h_t k_t d\sigma$$

and

$$|u_t(X)| \lesssim \left(\frac{r}{|X - Q_0|} \right)^\eta \frac{1}{\omega_t(\Delta_{|X-Q_0|}(Q_0))} \quad (3.21)$$

for some $\eta > 0$. We cover Γ_{R_0} by balls of radius $8R_0$ such that the balls of radius $2R_0$ are disjoint and do not intersect with $T(\Delta_{2R_0})$. Using Cacciopoli's inequality, the maximum principle and (3.21), we have

$$\left(\int_{B(Q_l, 2R_0)} |\nabla u_t|^2 dY \right)^{1/2} \lesssim R_0^{-1+n/2} \sup_{B(Q_l, 2R_0)} \lesssim R_0^{-1+n/2} \left(\frac{r}{R_0} \right)^\eta \frac{1}{\omega_t(\Delta_{R_0}(Q_0))}, \quad (3.22)$$

which shows that

$$\left(\int_{\Gamma_{R_0}} |\nabla u_t|^2 dY \right)^{1/2} \lesssim r^\eta \quad (3.23)$$

with constant depending also on $\text{diam}\Omega$ and R_0 . The claim follows by combining (3.19), (3.20) and (3.23).

Claim 3. For fixed $Q_0 \in \partial\Omega$, let $0 < r < R < 8R_0 < \frac{\delta(0)}{4}$. Then

$$\int_{T(\Delta_R) \setminus T(\Delta_r)} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \leq \sum_{j=1}^L \left(\int_{T(\Delta_{8^j r}) \setminus T(\Delta_{8^{j-1} r})} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \right) \lesssim \mathcal{C}(a) \quad (3.24)$$

where $\Delta_r = \Delta_r(Q_0)$, $\Delta_R = \Delta_R(Q_0)$ and L is chosen such that $8^L r \leq R < 8^{L+1} r$.

Proof of Claim 3. We will estimate

$$I_j = \int_{T(\Delta_{8^j r}) \setminus T(\Delta_{8^{j-1} r})} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY$$

for fixed j . We start by defining a dyadic decomposition on $A_j = T(\Delta_{8^j r}) \setminus T(\Delta_{8^{j-1} r})$. Cover $\Delta'_j = \Delta_{8^j r} \setminus \Delta_{8^{j-1} r}$ by balls $B_i(Q_i, \rho)$ with center $Q_i \in \Delta'_j$ and radius $\rho = 8^{j-6} r$. The numbers of the balls needed is roughly $c_n = 8^{7n-7} - 8^{4n-3}$. In that case we are able to cover small strips close to the boundary by balls. Then we split

$$A_j = \left[\left(\bigcup_{i=1}^{c_n} B_i \right) \cap A_j \right] \cup \left[A_j \setminus \bigcup_{i=1}^{c_n} B_i \right] = V_j \cup W_j.$$

Following the pattern in the proof of Lemma 7.7 in [14] we will estimate first the term close to the boundary

$$\int_{V_j} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY \leq \lim_{\epsilon \rightarrow 0} \int_{V_j \setminus (\partial\Omega, \epsilon)} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY = \lim_{\epsilon \rightarrow 0} I_j^\epsilon$$

and

$$\begin{aligned} I_j^\epsilon &\lesssim \sum_{\substack{Q_\alpha^k \in 3\Delta'_j \\ \text{diam} Q_\alpha^k \leq 8^j r}} \sup_{I_\alpha^k} |\varepsilon(Y)| \int_{I_\alpha^k} |\nabla G_t(0, Y)| |\nabla u_t(Y)| dY \\ &\lesssim \sum_{\substack{Q_\alpha^k \in 3\Delta'_j \\ \text{diam} Q_\alpha^k \leq 8^j r}} \left(\int_{I_\alpha^k} \frac{a^2(Y) G_t(0, Y)^2}{\delta(Y)^2} dY \right)^{1/2} \left(\int_{I_\alpha^k} |\nabla u_t|^2 dY \right)^{1/2}. \end{aligned} \quad (3.25)$$

Now for $Y \in I_\alpha^k$

$$G_t(0, Y) \sim \frac{\omega_t(Q_\alpha^k)}{(\text{diam} Q_\alpha^k)^{n-2}} \quad (3.26)$$

and

$$\int_{I_\alpha^k} |\nabla u|^2 dY \lesssim (\text{diam} Q_\alpha^k)^{-2} \int_{2I_\alpha^k} u_t^2 dY. \quad (3.27)$$

On the other hand, for $Y \in 2I_\alpha^k$

$$|u_t(Y)| \lesssim \left(\frac{\text{diam} Q_\alpha^k}{8^j r} \right)^\eta \sup_{A_j} |u_t| \quad (3.28)$$

for some $\eta > 0$. We will now estimate $\sup_{A_j} |u_t|$. In particular, for $Z \in A_j$ we have

$$\begin{aligned} |u_t(Z)| &\leq \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} \left| f - \oint_{\Delta_r} f d\omega_t \right| d\omega_t^Z \\ &\leq \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} |f| d\omega_t^Z + \frac{\omega_t^Z(\Delta_r)}{\omega_t(\Delta_r)^2} \int_{\Delta_r} |f| d\omega_t \end{aligned} \quad (3.29)$$

and

$$\int_{\Delta_r} |f| d\omega_t^Z = \int_{\Delta_r} K_t(Z, Q) |f| d\omega_t$$

where

$$K_t(Z, Q) \sim \frac{G_t(Z, A_r(Q_0))}{G_t(0, A_r(Q_0))} \sim \frac{\omega_t^Z(\Delta_r)}{\omega_t(\Delta_r)}.$$

Therefore (3.29) becomes

$$\begin{aligned} |u_t(Z)| &\lesssim \frac{\omega_t^Z(\Delta_r)}{\omega_t(\Delta_r)^2} \int_{\Delta_r} |f| d\omega_t \\ &\lesssim \sigma(\Delta_r) \frac{\omega_t^Z(\Delta_r)}{\omega_t(\Delta_r)^2} \left(\oint_{\Delta_r} |f| d\omega_t \right)^{1/2} \left(\oint_{\Delta_r} k_t^2 d\sigma \right)^{1/2} \\ &\lesssim \frac{\omega_t^Z(\Delta_r)}{\omega_t(\Delta_r)} \lesssim \left(\frac{\delta(Z)}{8^j r} \right)^\eta \frac{\omega_t^{P_j}(\Delta_r)}{\omega_t(\Delta_r)} \end{aligned} \quad (3.30)$$

for some $\eta > 0$ and $P_j \in W_j$. Now

$$\frac{\omega_t(\Delta_r)}{\omega_t(\Delta_{8^j r})} \sim \frac{\omega_t^{P_j}(\Delta_r)}{\omega_t^{P_j}(\Delta_{8^j r})}$$

thus from (3.30), we obtain

$$|u_t(Z)| \lesssim \left(\frac{\delta(Z)}{8^j r} \right)^\eta \frac{1}{\omega_t(\Delta_{8^j r})} \lesssim \frac{1}{\omega_t(\Delta_{8^j r})}$$

and (3.28) becomes

$$|u_t(Y)| \lesssim \left(\frac{\text{diam} Q_\alpha^k}{8^j r} \right)^\eta \frac{1}{\omega_t(\Delta_{8^j r})} \quad (3.31)$$

for $Y \in 2I_\alpha^k$, where for simplicity we used the same notation for the exponent $\eta > 0$.

We now return to the estimate of I_j^ε in (3.25) to obtain

$$\begin{aligned}
I_j^\varepsilon &\lesssim \sum_{\substack{Q_\alpha^k \in 3\Delta'_j \\ \text{diam} Q_\alpha^k \leq 8^j r}} \left(\int_{I_\alpha^k} \frac{a^2(Y)}{\delta(Y)} dY \right)^{1/2} \left(\frac{1}{\text{diam} Q_\alpha^k} \right)^{1/2+1} \frac{\omega_t(Q_\alpha^k)}{(\text{diam} Q_\alpha^k)^{n-2}} \\
&\quad \cdot \left(\frac{\text{diam} Q_\alpha^k}{8^j r} \right)^\eta \frac{1}{\omega_t(\Delta_{8^j r})} (\text{diam} Q_\alpha^k)^{n/2} \\
&\lesssim \left(\frac{1}{8^j r} \right)^\eta \frac{1}{\omega_t(\Delta_{8^j r})} \sum_{\substack{Q_\alpha^k \in 3\Delta'_j \\ \text{diam} Q_\alpha^k \leq 8^j r}} \left(\frac{1}{\sigma(Q_\alpha^k)} \int_{I_\alpha^k} \frac{a^2(Y)}{\delta(Y)} dY \right)^{1/2} \omega_t(Q_\alpha^k) (\text{diam} Q_\alpha^k)^\eta \\
&\lesssim \left(\frac{1}{8^j r} \right)^\eta \frac{1}{\omega_t(\Delta_{8^j r})} \mathcal{C}(a) \sum_{\substack{Q_\alpha^k \in 3\Delta'_j \\ \text{diam} Q_\alpha^k \leq 8^j r}} \omega_t(Q_\alpha^k) (\text{diam} Q_\alpha^k)^\eta \\
&\lesssim 8^{-2j\eta} \mathcal{C}(a). \tag{3.32}
\end{aligned}$$

Finally to estimate the integral over W_j , we cover W_j with balls B_{jl} with centers $Q_{jl} \in W_j$ and radius $\rho_j = 8^{j-11}r$. Following the pattern in the proof above we have that for $Y \in B_{jl}$

$$G_t(0, Y) \sim \frac{\omega_t(\Delta_{jl})}{(8^j r)^{n-2}}$$

and

$$\left(\int_{B_{jl}} |\nabla u_t|^2 dY \right)^{1/2} \leq (8^j r)^{\frac{n-2}{2}} \left(\frac{r}{8^j r} \right)^\eta \frac{1}{\omega_t(\Delta_{jl})}.$$

Therefore,

$$\begin{aligned}
\int_{W_j} |\varepsilon(Y)| |\nabla_Y G_t| |\nabla u_t| dY &\lesssim \sum_l \sup_{B_{jl}} |\varepsilon(Y)| \left(\int_{B_{jl}} \frac{G_t(0, Y)^2}{\delta(Y)^2} dY \right)^{1/2} \left(\int_{B_{jl}} |\nabla u_t|^2 dY \right)^{1/2} \\
&\lesssim \sum_l \left(\int_{B_{jl}} \frac{a^2(Y) G_t(0, Y)^2}{\delta(Y)^2} dY \right)^{1/2} \left(\int_{B_{jl}} |\nabla u_t|^2 dY \right)^{1/2} \\
&\lesssim \mathcal{C}(a) \sum_l \frac{1}{(8^j r)^{1/2}} \frac{\omega_t(\Delta_{jl})}{(8^j r)^{n-2}} (8^j r)^{\frac{n}{2}-1} 8^{-j\eta} \frac{1}{\omega_t(\Delta_{jl})} (8^j r)^{\frac{n-1}{2}} \\
&\lesssim 8^{-\eta j} \mathcal{C}(a) \tag{3.33}
\end{aligned}$$

for some $\eta > 0$, which concludes the proof of Claim 3.

To finish the proof of Lemma 3.4, we write

$$\Omega = T(\Delta_{Mr}) \cup \left(\Omega \setminus T(\Delta_{r^\beta}) \right) \cup \left(T(\Delta_{r^\beta}) \setminus T(\Delta_{Mr}) \right)$$

and combine Claims 1, 2 and 3. □

Theorem 3.5. *Let Ω be a CAD, there exist $\eta_0 > 0$ and $q_0 > 2$ such that if $\mathcal{C}(a) < \eta_0$ and $\omega_0 \in B_{q_0}(\sigma)$ then*

$$\left(\int_{\Delta_r} k_1^2 d\sigma \right)^{1/2} \leq \left[\frac{(f_{\Delta_r} k_0^2 d\sigma)^{1/2}}{f_{\Delta_r} k_0 d\sigma} + Cr^\gamma + C \sup_{\substack{Q \in \partial\Omega \\ s \leq r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))} \int_{T(\Delta_s(Q))} \frac{\alpha(Y)^2}{\delta(Y)} dY \right)^{1/2} \right] \int_{\Delta_r} k_1 d\sigma \quad (3.34)$$

where the positive constants β , γ and C only depend on the CAD constants, the ellipticity constants and the dimension.

Proof. By (3.15) and the fundamental theorem of calculus we have

$$\Psi(1) \leq \Psi(0) + e(r)$$

where

$$e(r) = C \left(r^\gamma + \sup_{\substack{Q \in \partial\Omega \\ s \leq r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))} \int_{T(\Delta_s(Q))} \frac{\alpha(Y)^2}{\delta(Y)} dY \right)^{1/2} \right)$$

and

$$\Psi(s) = \frac{1}{\omega_s(\Delta_r)} \int_{\Delta_r} f k_s d\sigma$$

for $f \geq 0$, $\text{spt} f \subset \Delta_r$ and $\|f\|_{L^2(d\sigma/\Delta_r)} \leq 1$. By the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} \int_{\Delta_r} f k_1 d\sigma &\leq \left[\frac{1}{\omega_0(\Delta_r)} \int_{\Delta_r} f k_0 d\sigma + e(r) \right] \int_{\Delta_r} k_1 d\sigma \\ &\leq \left[\frac{1}{\omega_0(\Delta_r)} \left(\int_{\Delta_r} f^2 d\sigma \right)^{1/2} \left(\int_{\Delta_r} k_0^2 d\sigma \right)^{1/2} + e(r) \right] \int_{\Delta_r} k_1 d\sigma \\ &\leq \left[\frac{(f_{\Delta_r} k_0^2 d\sigma)^{1/2}}{f_{\Delta_r} k_0 d\sigma} + e(r) \right] \int_{\Delta_r} k_1 d\sigma. \end{aligned}$$

By duality we have

$$\left(\int_{\Delta_r} k_1^2 d\sigma \right)^{1/2} \leq \left[\frac{(f_{\Delta_r} k_0^2 d\sigma)^{1/2}}{f_{\Delta_r} k_0 d\sigma} + e(r) \right] \int_{\Delta_r} k_1 d\sigma$$

and the proof is complete. \square

The following result is an easy corollary of Theorem 3.5

Corollary 3.6. *Let Ω be a CAD. Assume that $\log k_0 \in VMO(\sigma)$ and that L_1 is a perturbation of L_0 whose deviation from L_0 has vanishing Carleson constant, then given $\varepsilon > 0$ there exists $r_0 > 0$ such that for every $r \leq r_0$*

$$\left(\int_{\Delta_r} k_1^2 d\sigma \right)^{1/2} \leq (1 + \varepsilon) \int_{\Delta_r} k_1 d\sigma.$$

In [13] the authors proved that the logarithm of the Poisson kernel on a chord arc domain with vanishing constant belongs to $VMO(\sigma)$. Thus Corollary 3.6 yields:

Corollary 3.7. *If Ω is a chord arc domain with vanishing constant and L_1 is a perturbation of the Laplacian whose deviation from the Laplacian has vanishing Carleson constant then $\log k_1 \in VMO(\partial\Omega)$.*

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